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CALCULUS OF VARIATIONS IN MEAN AND CONVEX LAGRANGIANS, III

BY

JOËL BLOT

Faculté des Sciences de Limoges, Université de Limoges, Département de Mathématiques, 123 Avenue Albert Thomas, 87060 Limoges Cédex, France

ABSTRACT

For the almost periodic or periodic solutions of an Euler–Lagrange equation, with a convex lagrangian, under a condition of symmetry on the lagrangian, we establish a necessary condition that involves the second differential of the lagrangian. We deduce from this some results of non-existence.

From a lagrangian $L \in C^1(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R})$ one can formulate an Euler–Lagrange equation:

$$\text{(E.L.)} \qquad L_x(x, \dot{x}) = \frac{d}{dt} L_{\dot{x}}(x, \dot{x}).$$

We are interested in the periodic or u.a.p. (uniformly almost periodic) solutions of (E.L.).

$\mathrm{AP}^k(\mathbf{R}^n)$ denotes the space of u.a.p. functions from $\mathbf{R}$ into $\mathbf{R}^n$ of which the derivatives, until order k, are also u.a.p. For $x \in \mathrm{AP}^0(\mathbf{R}^n)$,

$$\mathcal{M}\{x\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

denotes the mean value of x, and for $\lambda \in \mathbf{R}$, $a(x;\lambda) := \mathcal{M}\{x(t)e^{-i\lambda t}\}_t$.

In [2], [3], [4], we built a functional

$$\phi(x) := \mathcal{M}\{L(x, \dot{x})\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} L(x(t), \dot{x}(t))dt$$

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on $AP^1(\mathbf{R}^n)$ and showed that $\phi'(x)=0$ if and only if x is a u.a.p. solution of (E.L.). If $C^1_T(\mathbf{R}^n)$ denotes the space of T-periodic continuously derivable functions with values in $\mathbf{R}^n$, the restriction of ϕ at $C^1_T(\mathbf{R}^n)$ is

$$\phi_T(x):=\frac{1}{T}\int_0^T L(x(t),\dot{x}(t))dt;$$

and $\phi'_T(x)=0$ if and only if x is a T-periodic solution of (E.L.).

In this paper, we consider the particular case where L is a convex function. The convexity of L implies the convexity of ϕ, and $\phi'(x)=0$ if and only if $\phi(x)=\operatorname{Min}\phi$. Furthermore Arg min ϕ, that is, the set of the $x\in AP^1(\mathbf{R}^n)$ that minimize ϕ, is exactly the set of the u.a.p. solutions of (E.L.), and is a convex subset of $AP^1(\mathbf{R}^n)$.

THEOREM 1. *Let $L\in C^2(\mathbf{R}^n\times\mathbf{R}^n,\mathbf{R})$ be a convex lagrangian. Let $x\in AP^1(\mathbf{R}^n)$ a u.a.p. (or periodic) solution of (E.L.); $c:=\mathcal{M}\{x\}$. We assume:*

(Sym.) $$L_{x\dot{x}}(c,0)=L_{\dot{x}x}(c,0).$$

Then we have

$$x(\mathbf{R})-c\subset \operatorname{Ker} L_{xx}(c,0)\cap \operatorname{Ker} L_{\dot{x}\dot{x}}(c,0).$$

COMMENTS. The condition of symmetry (Sym) is satisfied, for example, when $n=1$ since L is of class C^2, or when the state variable and the rate variable are separated: $L(x,\dot{x})=V(x)+W(\dot{x})$, or, more generally, when $L(x,\dot{x})=V(x)+W(\dot{x})+B(x,\dot{x})$ with B bilinear and symmetric.

LEMMA. *Let A and B be two real symmetric $n\times n$ non-negative definite matrices. Let $u\in AP^1(\mathbf{R}^n)$ such that $\mathcal{M}\{u\}=0$ and $B\dot{u}\in AP^1(\mathbf{R}^n)$.*

Then the two following assertions are equivalent:

(i) *for all $t\in\mathbf{R}$,* $Au(t)=(d/dt)(B\dot{u}(t))$,

(ii) $u(\mathbf{R})\subset \operatorname{Ker} A\cap \operatorname{Ker} B$.

PROOF OF LEMMA. It is clear that (ii) implies (i) since, under (ii), (i) is reduced to $0=0$. Conversely, we assume that (i) is verified. Then, for all $\lambda\in\mathbf{R}$,

$$a(Au;\lambda)=a\left(\frac{d}{dt}(B\dot{u}),\lambda\right).$$

We have $a(Au;\lambda)=Aa(u;\lambda)$ and

$$
\begin{aligned}
a\left(\frac{d}{dt}(B\mathring{u}),\lambda\right) &= \mathcal{M}\left\{\frac{d}{dt}(B\mathring{u})\cdot e^{-i\lambda t}\right\}_t \\
&= -\mathcal{M}\left\{B\mathring{u}(t)\cdot\frac{d}{dt}e^{-i\lambda t}\right\}_t \quad \text{(cf. [3] Prop. 2)} \\
&= i\lambda\mathcal{M}\{B\mathring{u}(t)\cdot e^{-i\lambda t}\}_t = i\lambda B\mathcal{M}\{\mathring{u}(t)e^{-i\lambda t}\}_t \\
&= i\lambda B\left(-\mathcal{M}\left\{u(t)\frac{d}{dt}e^{-i\lambda t}\right\}_t\right) = (i\lambda)^2 Ba(u,\lambda) = -\lambda^2 Ba(u;\lambda).
\end{aligned}
$$

And so we obtain, for all $\lambda \in \mathbf{R}$, $(A + \lambda^2 B)a(u;\lambda) = 0$, i.e. $(A + \lambda^2 B)\operatorname{Re} a(u;\lambda) = 0$ and $(A + \lambda^2 B)\operatorname{Im} a(u;\lambda) = 0$. We remark that if $v \in \mathbf{R}^n$ verifies $(A + \lambda^2 B)v = 0$, then

$$0 = \langle (A + \lambda^2 B)v, v\rangle = \langle Av, v\rangle + \lambda^2\langle Bv, v\rangle = 0,$$

therefore, when $\lambda \neq 0$, $\langle Av, v\rangle = 0$ and $\langle Bv, v\rangle = 0$, therefore $Av = 0$ and $Bv = 0$, i.e. $v \in \operatorname{Ker} A \cap \operatorname{Ker} B$.

Consequently, for all $\lambda \in \mathbf{R} \mid \{0\}$, $\operatorname{Re} a(u;\lambda)$ and $\operatorname{Im} a(u;\lambda)$ are into $\operatorname{Ker} A \cap \operatorname{Ker} B$. Introduce now P, the orthogonal projector on $\operatorname{Ker} A \cap \operatorname{Ker} B$. We easily verify that $\operatorname{Re} a(Pu;\lambda) = P(\operatorname{Re} a(u;\lambda)) = \operatorname{Re} a(u;\lambda)$, and $\operatorname{Im} a(Pu;\lambda) = P(\operatorname{Im} a(u;\lambda)) = \operatorname{Im} a(u;\lambda)$. Therefore for all $\lambda \in \mathbf{R}$, $a(Pu;\lambda) = a(u;\lambda)$ and, by the Uniqueness Theorem of Fourier–Bohr series (cf. [1], p. 27), we obtain that $P(u) = u$. This equality means that $u(\mathbf{R}) \subset \operatorname{Ker} A \cap \operatorname{Ker} B$. □

PROOF OF THEOREM 1. We denote $u := x - c$. We have shown in [3], [4] that c is also a solution of (E.L.). Since Arg min ϕ is a convex set, we have, for all $\theta \in [0, 1]$, $c + \theta u = (1 - \theta)c + \theta(c + u) \in \operatorname{Arg\,min} \phi$. Therefore, for all $h \in AP^1(\mathbf{R}^n)$ and all $\theta \in [0, 1]$, we have:

$$0 = \phi'(c + \theta u)\cdot h = \mathcal{M}\{\langle L_x(c + \theta u, \theta\mathring{u}), h\rangle + \langle L_{\dot{x}}(c + \theta u, \theta\mathring{u}), \mathring{h}\rangle\},$$

and so

$$
\begin{aligned}
0 &= \frac{d}{d\theta}\phi'(c + \theta u)\cdot h\,\Big|_{\theta=0} \\
&= \mathcal{M}\{\langle L_{xx}(c, 0)u, h\rangle + \langle L_{x\dot{x}}(c, 0)\mathring{u}, h\rangle + \langle L_{\dot{x}x}(c, 0)u, h\rangle \\
&\qquad + \langle L_{\dot{x}\dot{x}}(c, 0)\mathring{u}, \mathring{h}\rangle\}.
\end{aligned}
$$

As a consequence of the hypothesis (Sym.) we have

$$\langle L_{x\dot{x}}(c,0)\dot{u}, h\rangle + \langle L_{\dot{x}x}(c,0)u, \dot{h}\rangle = \frac{d}{dt}\langle L_{x\dot{x}}(c,0)u, h\rangle.$$

Therefore (cf. [3] Prop. 1);

$$\mathcal{M}\{\langle L_{x\dot{x}}(c,0)\dot{u}, h\rangle + \langle L_{\dot{x}x}(c,0)u, \dot{h}\rangle\} = 0.$$

This remark permits us to obtain, for all $h \in \mathrm{AP}^1(\mathbf{R}^n)$,

$$\mathcal{M}\{\langle L_{xx}(c,0)u, h\rangle + \langle L_{\dot{x}\dot{x}}(c,0)\dot{u}, \dot{h}\rangle\} = 0. \tag{1}$$

But the left side of (1) is exactly the differential of the quadratic functional

$$Q(z) := \tfrac{1}{2}\mathcal{M}\{\langle L_{xx}(c,0)z, z\rangle + \langle L_{\dot{x}\dot{x}}(c,0)\dot{z}, \dot{z}\rangle\},$$

at u in the direction h : $Q'(u).h = 0$. Then the Dubois–Reymond Lemma (cf. [3] Part B) implies that $L_{\dot{x}\dot{x}}(c,0)\dot{u} \in \mathrm{AP}^1(\mathbf{R}^n)$ and:

$$L_{xx}(c,0)u = \frac{d}{dt}(L_{\dot{x}\dot{x}}(c,0)\dot{u}). \tag{2}$$

In taking $A = L_{xx}(c,0)$ and $B = L_{\dot{x}\dot{x}}(c,0)$, the preceding lemma allows us to conclude that:

$$x(\mathbf{R}) - c = u(\mathbf{R}) \subset \operatorname{Ker} L_{xx}(c,0) \cap \operatorname{Ker} L_{\dot{x}\dot{x}}(c,0). \tag{3}$$ □

We can translate the conditions (2) and (3) into a condition on the functional ϕ:

$$\phi''(c)(u,h) = \frac{d}{d\theta}\phi'(c+\theta u)h\,\Big|_{\theta=0} = 0,$$

and so each one of the conditions (2) and (3) is equivalent to

$$\phi''(c)(u,.) = 0. \tag{4}$$

And so the non-constancy of $x = c + u$ implies the degeneration of the bilinear form $\phi''(c)$.

Corollary. *We suppose that L is a convex function of class C^2. Let $c \in \mathbf{R}^n$ such that $L_x(c,0) = 0$. We assume the condition*

$$L_{x\dot{x}}(c,0) = L_{\dot{x}x}(c,0). \tag{Sym.}$$

Then we have:

(i) *If* $\operatorname{Ker} L_{xx}(c,0) \cap \operatorname{Ker} L_{\dot{x}\dot{x}}(c,0) = \{0\}$ *then* (E.L.) *does not admit any non-constant periodic or u.a.p. solution with a mean value equal to c.*

(ii) *If* $L_{\dot{x}\dot{x}}(c,0)$ *is positive definite, then the alone u.a.p. solution of (E.L.) with a mean value equal to c is the constant c.*

(iii) *If* $L_{xx}(c,0)$ *is positive definite, then the constant c is the unique u.a.p. solution of (E.L.).*

PROOF. (i) results from Theorem 1, and (ii) results from (i). For (iii), the assertion (i) implies that the unique u.a.p. solution of (E.L.), with a mean value equal to c, is c. But we have shown in [5] (Prop. 2) that the condition $L_{xx}(c,0)$ implies that all the u.a.p. solutions of (E.L.) admit c as mean value. □

In order to write the canonical form of (E.L.) one uses the hamiltonian

$$H(x,p) := \sup\{\langle p, y\rangle - L(x,y) \mid y \in \mathbf{R}^n\}$$

which is a concave–convex function. When we want H to be of class C^1, generally we assume that $L_{\dot{x}\dot{x}}(x,y)$ is invertible, therefore $L_{\dot{x}\dot{x}}(x,y) > 0$ since L is convex. But, due to the assertion (ii) of Corollary, when $L_{x\dot{x}} = L_{\dot{x}x}$ and when $L_{\dot{x}\dot{x}} > 0$, (E.L.) cannot possess any non-constant u.a.p. solution. And so, when $L_{x\dot{x}} = L_{\dot{x}x}$, generally, if (E.L.) possesses non-constant u.a.p. solutions, we cannot associate to (E.L.) the Hamiltonian System: $\dot{x} = H_p(x,p)$, $\dot{p} = -H_x(x,p)$, but we can only associate to it the Hamiltonian inclusion: $\dot{x} \in \partial p\, H(x,p)$, $-\dot{p} \in \partial x\, H(x,p)$ (cf. [6] §1.4, and [9]).

Another consequence of the assertion (ii) of Corollary is that a second order differential equation: $\ddot{x} = V'(x)$, where V is a convex function, cannot admit any non-constant u.a.p. (or periodic) trajectories, since it is the Euler–Lagrange equation associated at the lagrangian $L(x,\dot{x}) = \frac{1}{2}|\dot{x}|^2 + V(x)$, and, for all $c \in \mathbf{R}^n$, we have $L_{x\dot{x}}(c,0) = L_{\dot{x}x}(c,0) = 0$, $L_{xx}(c,0) = I > 0$.

Rockafellar ([9]) and Gaines and Peterson ([8]) consider, for a fixed real $T > 0$, and for a convex lagrangian L, the variational problem:

$$\text{(5)} \qquad \text{Minimize} \int_0^T L(x(t), \dot{x}(t))\,dt, \qquad x \in H_T^1(\mathbf{R}^n)$$

where

$$H_T^1(\mathbf{R}^n) = \{z \in H_{\mathrm{loc}}^1(\mathbf{R}, \mathbf{R}^n) \mid z \text{ is } T\text{-periodic}\}$$
$$= H^1(\mathbf{R}\,|_{T\mathbf{Z}}, \mathbf{R}^n).$$

The problem (5) is equivalent to

$$\text{(6)} \qquad \text{Minimize } \phi_T(x), \qquad x \in H_T^1(\mathbf{R}^n).$$

The necessary condition of the first order for (5) or (6) is

$$(\mathrm{E.L.})_{\mathrm{W}} \qquad L_x(x(t), \dot{x}(t)) = \frac{d}{dt} L_{\dot{x}}(x(t), \dot{x}(t)) \quad \text{Lebesgue-a.e. on } \mathbf{R}.$$

That constitutes a notion of weak (T-periodic) solution of (E.L.).

Theorem 2. *Let $L \in C^1(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R})$ be a convex lagrangian. Let T be a positive real number. We assume that $\phi_T \in C^1(H_T^1(\mathbf{R}^n), \mathbf{R})$. Let $x \in H_T^1(\mathbf{R}^n)$ be a solution of $(E.L.)_W$ Then*:

(i) *$c := (1/T)\int_0^T x(t)dt$ is a solution of $(E.L.)_W$, and also of $(E.L.)$.*

(ii) *If L is of class C^2, if ϕ_T is twice Gâteaux-differentiable and if $L_{x\dot{x}}(c, 0) = L_{\dot{x}x}(c, 0)$, then $x(\mathbf{R}) - c \subset \operatorname{Ker} L_{xx}(c, 0) \cap \operatorname{Ker} L_{\dot{x}\dot{x}}(c, 0)$.*

Proof. The convexity of L implies the convexity of ϕ_T, and consequently, $\phi_T(x) = \operatorname{Min} \phi_T$ if and only if $\phi_T'(x) = 0$ if and only if x is a T-periodic solution of $(\mathrm{E.L.})_{\mathrm{W}}$. π denotes the orthogonal projector from $H_T^1(\mathbf{R}^n)$ onto $\mathbf{R}^n$ identified with the constant functions from $\mathbf{R}$ into $\mathbf{R}^n$. In using the Fourier series of x:

$$x(t) = c + \sum_{k \geqq 1} \left(a_k \cos \frac{2\pi}{T} kt + b_k \sin \frac{2\pi}{T} kt \right),$$

we can verify that, for all $c_1 \in \mathbf{R}^n$, we have

$$\int_0^T \langle x(t) - c, c_1 \rangle dt = 0 \quad \text{and} \quad \int_0^T \langle \dot{x}(t) - \dot{c}, \dot{c}_1 \rangle dt = 0.$$

And so $x - c$ is orthogonal to $\mathbf{R}^n$ in $H_T^1(\mathbf{R}^n)$, therefore $c = \pi(x)$.

Let S be a real positive number such that S/T is irrational. We denote $\tau_S f(t) := f(t + S)$, then τ_S is a bounded linear operator from $H_T^1(\mathbf{R}^n)$ into itself, and the set of the elements of $H_T^1(\mathbf{R}^n)$ that are invariant by τ_S is exactly $\mathbf{R}^n$. Since L is autonomous, we have $\phi_T \circ \tau_{kS} = \phi_T$ for all $k \in \mathbf{N}$, and so $\tau_{kS} x \in \operatorname{Arg\,min} \phi_T$. Moreover, $\operatorname{Arg\,min} \phi_T$ is a closed convex subset of $H_T^1(\mathbf{R}^n)$.

By the Ergodic Theorem of Von Neumann (cf. [7], p. 34)

$$\pi(x) = \lim_{v\to\infty} \frac{1}{v} \sum_{k=0}^{v-1} \tau_{kS} x \qquad \text{in } H_T^1(\mathbf{R}^n).$$

We remark that

$$\frac{1}{v} \sum_{k=0}^{v-1} \tau_{kS} x = \sum_{k=0}^{v-1} \frac{1}{v} \tau_{kS} x$$

is a convex combination of elements of Arg min ϕ_T, and therefore $c = \pi(x) \in$ Arg min ϕ_T. That justifies (i).

We denote $u := x - c$. By the same arguments as those of the proof of Theorem 1, we verify that, for all $\theta \in [0, 1]$ and all $h \in H_T^1(\mathbf{R}^n)$, $\phi_T'(c + \theta u) . h = 0$, and

$$\begin{aligned} 0 &= \frac{d}{d\theta} \phi_T'(c + \theta u) . h \,\Big|_{\theta=0} \\ &= \int_0^T \{\langle L_{xx}(c, 0)u, h\rangle + \langle L_{\dot{x}\dot{x}}(c, 0)\dot{u}, \dot{h}\rangle\} dt. \end{aligned}$$

That implies

$$L_{xx}(c, 0) = \frac{d}{dt} L_{\dot{x}\dot{x}}(c, 0)\dot{u} \qquad \text{Lebesgue-a.e.}$$

After that, in working with the Fourier series, we obtain: Re $a(u; (2\pi/T)k)$ and Im $a(u; (2\pi/T)k)$ belong to Ker $L_{xx}(c, 0) \cap$ Ker $L_{\dot{x}\dot{x}}(c, 0)$. Then $P(u)$ and u possess the same Fourier series, therefore $P(u) = u$, and $u(\mathbf{R}) \subset$ Ker $L_{xx}(c, 0) \cap$ Ker $L_{\dot{x}\dot{x}}(c, 0)$. □

The consequences of Theorem 2 are similar to those of Theorem 1. For example, under the hypothesis $L_{x\dot{x}}(c, 0) = L_{\dot{x}x}(c, 0)$, the condition Ker $L_{xx}(c, 0) \cap$ Ker $L_{\dot{x}\dot{x}}(c, 0) = \{0\}$ forbids the existence of non-constant weak T-periodic solutions with c as mean value, of Euler–Lagrange's equation, or equivalently of the problem of minimization (5) under the constraint $(1/T) \int_0^T x(t) dt = c$.

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